

The Dominated Integral

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I. INTRODUCTION AND STATEMENT OF RESULTS

In this paper we introduce a new concept of integral on $(0, 1]$ (the "dominated integral") intimately connected with the problem of numerical integration of unbounded functions. The existence of the dominated integral I of a function f implies the convergence of the improper Riemann integral $\int_{0+}^1 f(x) dx$, and its equality to I , but $\int_{0+}^1 f(x) dx$ may converge without existence of I . An important difference between the two concepts is that while $\int_{0+}^1 f(x) dx$ is defined as an iterated limit (i.e., the limit of a proper Riemann integral, itself a limit), the dominated integral is defined as a single limit.

Recently, a concept similar to that of the dominated integral was introduced, concerning integration over $[0, \infty)$ (the "simple integral," see [1, 2]). However, the dominated integral and the simple integral appear to be of somewhat different nature: The first is an absolute integral, the second is not. In fact, one can readily see that if one tries to imitate the definition of simple integrability by replacing $[0, \infty)$ with $(0, 1]$, and ∞ by 0, one obtains essentially the concept of (proper) Riemann integrability on $[0, 1]$.

The theory of the dominated integral is strongly related to the problem: Under what conditions can quadrature formulas effective for Riemann integrable functions on $[0, 1]$ be used for the numerical evaluation of improper Riemann integrals $\int_{0+}^1 f(x) dx$? The theoretical study of this type of question was initiated by Davis and Rabinowitz [3], and was followed by further work [4-7]. As the practical use of quadrature formulas to compute improper Riemann integrals without a theoretical justification has become quite common, the need for such a theoretical study is unquestionable.

It turns out that, for a function f on $(0, 1]$, the existence of its dominated

integral is a necessary and sufficient condition for f to be improperly Riemann integrable there, and to satisfy $\lim_{n \rightarrow \infty} \Phi_n^*(f) = \int_{0+}^1 f(x) dx$ for every sequence $(\Phi_n^*)_{n=1}^\infty$ of quadrature formulas of a very general type. This is shown in a subsequent article [8]. Here we only mention two applications from [8]:

I. Suppose $(R_n(f))_{n=1}^\infty$ is a sequence of compound rules on $[0, 1]$ not involving $f(0)$, and integrating 1 exactly, namely,

$$R_n(f) \equiv \sum_{k=1}^n \sum_{r=1}^m w_r n^{-1} f((k-1+x_r)n^{-1}), \quad n = 1, 2, \dots,$$

where w_1, \dots, w_m are given complex constants with $\sum_{j=1}^m w_j = 1$, and $0 < x_1 < \dots < x_m \leq 1$. Then $\lim_{n \rightarrow \infty} R_n(f) = \int_{0+}^1 f(x) dx$ for every f whose dominated integral exists.

II. One can define the dominated integral on any interval $(a, b]$, $-\infty < a < b < \infty$. Let $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$, $-\frac{1}{2} \leq \beta \leq \frac{1}{2}$, and, for $n = 1, 2, \dots$, let

$$Q_n(F) \equiv \sum_{k=1}^n w_{n,k} F(x_{n,k})$$

be the n -point Gauss-Jacobi quadrature formula corresponding to the weight function $w(x) \equiv (1-x)^\alpha(1+x)^\beta$. If the dominated integral of a function f on $(-1, 1]$ exists, then $\lim_{n \rightarrow \infty} Q_n(f/w) = \int_{-1+}^1 f(x) dx$.

We now define the dominated integral, state its fundamental properties and relate it to the "simple integral" of [1, 2].

DEFINITION 1. Let f be a complex function on $(0, 1]$. A dominated integral of f is a number $I(f)$ having the property: For each $\epsilon > 0$ there exist δ and χ , $0 < \delta < 1$, $0 < \chi < 1$, such that

$$\left| I(f) - \sum_{j=1}^n f(\tau_j)(t_j - t_{j-1}) \right| < \epsilon \quad (1)$$

whenever $0 < t_0 < t_1 < \dots < t_n = 1$, $t_0 < \chi$, $t_{j-1} \leq \tau_j \leq t_j$, and $t_{j-1}t_j^{-1} > 1 - \delta$, $j = 1, 2, \dots, n$. (The justification for the adjective "dominated" will be clear from Theorem 3.) Dominant integrability of f means existence of such an $I(f)$. It is not difficult to see that if an $I(f)$ exists, it is unique.

THEOREM 1. If a dominated integral of f exists, then f is improperly Riemann integrable on $(0, 1]$, and $I(f) = \int_{0+}^1 f(t) dt$, the improper Riemann integral of f on $(0, 1]$.

(By the improper Riemann integral of $f(t)$ on $(0, 1]$ [$g(x)$ on $[0, \infty)$] we mean $\lim_{\epsilon \rightarrow 0+} \int_\epsilon^1 f(t) dt$ [$\lim_{R \rightarrow \infty} \int_0^R g(x) dx$], assumed to be finite, where for

each ϵ , $0 < \epsilon < 1$, $[R, R > 0]$ $\int_{\epsilon}^1 f(t) dt [\int_0^R g(x) dx]$ is assumed to exist as a proper Riemann integral.)

If f is a bounded complex function on $[a, b]$ we shall denote by $w(f, a, b)$ the oscillation of f on $[a, b]$, i.e., the supremum of the set of all $|f(t_1) - f(t_2)|$ with $a \leq t_1 \leq t_2 \leq b$.

Given a complex function $f(t)$ defined and bounded on each closed subinterval of $(0, 1]$, and any sequence $0 < t_0 < t_1 < \dots < t_n = 1$, let $OS(f; t_0, \dots, t_n)$ denote the oscillation sum

$$\sum_{j=1}^n w(f, t_{j-1}, t_j)(t_j - t_{j-1}).$$

DEFINITION 2. A complex function f satisfies the Riemann condition for the dominated integral (RCDI) if and only if the following two conditions hold:

- (i) f is defined on $(0, 1]$, and bounded on each of its closed subintervals; and
- (ii) for each $\epsilon > 0$ there exists δ , $0 < \delta < 1$, such that whenever $0 < t_0 < t_1 < \dots < t_n = 1$, and $t_{j-1}t_j^{-1} > 1 - \delta$, $j = 1, \dots, n$, we have $OS(f; t_0, \dots, t_n) < \epsilon$.

THEOREM 2. Let f be a complex function on $(0, 1]$. The dominated integral of f exists if and only if f satisfies RCDI.

COROLLARY 1. If the dominated integral of f exists, then so does the dominated integral of $|f|$.

Proof of Corollary 1. Since always $||f(t_1)| - |f(t_2)|| \leq |f(t_1) - f(t_2)|$, we see that if f satisfies RCDI, so does $|f|$.

COROLLARY 2. If the dominated integral of f exists, then there is a real monotone decreasing (by which we always mean "nonincreasing") function \hat{f} on $(0, 1]$ such that (i) $\hat{f} \geq |f|$ throughout $(0, 1]$, and (ii) \hat{f} is improperly Riemann integrable on $(0, 1]$.

Proof of Corollary 2. Set $\hat{f}(t) = \sup_{t \leq x \leq 1} |f(x)|$ ($0 < t \leq 1$). (See Theorem 2 and Definition 2, (i)). Then throughout $(0, 1]$, $\hat{f} \geq |f|$, and \hat{f} is monotone decreasing. Clearly $\hat{f}(t)$ is bounded on each closed subinterval of $(0, 1]$. We shall show that if $0 < a < b \leq 1$, then $w(\hat{f}, a, b) \leq w(|f|, a, b)$. It would then follow that \hat{f} satisfies RCDI, and we would be through by

Theorem 2. We need *only consider the case* $f(a) - f(b) > 0$. In this case $f(a) = \sup_{a \leq x \leq b} |f(x)|$. If $a \leq x_1 \leq x_2 \leq b$, then

$$\begin{aligned} |f(x_1) - f(x_2)| &= f(x_1) - f(x_2) \\ &\leq f(a) - f(b) \leq f(a) - |f(b)| = \sup_{a \leq x \leq b} (|f(x)| - |f(b)|) \\ &\leq \sup_{a \leq x \leq x' \leq b} ||f(x)| - |f(x')|| = w(|f|, a, b). \end{aligned}$$

DEFINITION 3. A complex function f on $(0, 1]([0, \infty))$ is called absolutely dominantly integrable (absolutely simply integrable) if and only if f is Riemann integrable on each closed bounded subinterval of $(0, 1]([0, \infty))$, and $|f|$ is dominantly integrable (simply integrable).

Observe that simple integrability implies improper Riemann integrability on $[0, \infty)$ [2, p. 931].

DEFINITION 4. A complex function f is said to have property D (for "dominated") on $(0, 1]$, or on $[0, \infty)$, if and only if f is Riemann integrable on each closed bounded subinterval of $(0, 1]$, or of $[0, \infty)$, and if there exists a monotone decreasing improperly Riemann integrable function g on $(0, 1]$, or on $[0, \infty)$, such that at each point of the interval, $g(t) \geq |f(t)|$.

THEOREM 3. *The following are equivalent: (i) dominant integrability; (ii) absolute dominant integrability; (iii) property D on $(0, 1]$; and (iv) Riemann integrability on each closed subinterval of $(0, 1]$ along with domination of absolute value on $(0, 1]$ by some dominantly integrable function.*

(That (i) implies (ii) follows from Theorem 1 and Corollary 1. That (ii) implies (iii) follows from Corollary 2. By Corollary 2 applied to the dominating function, we see that (iv) implies (iii). That (i) implies (iv) is seen by Theorem 1 and Corollary 1, letting the absolute value of the function dominate itself. Thus it merely remains to prove that (iii) implies (i).)

THEOREM 4. *Absolute simple integrability implies but is not implied by simple integrability and is implied by but does not imply property D on $[0, \infty)$.*

THEOREM 5. *Dominant integrability of a function f is equivalent to absolute simple integrability of $f(e^{-x})e^{-x}$, but does not imply property D of $f(e^{-x})e^{-x}$ on $[0, \infty)$.*

Let G denote the set of all real functions g with domain $[0, \infty)$, with g' continuous and negative on $[0, \infty)$ ($g'(0)$ being a right-hand derivative), $g(0) = 1$, and $\lim_{x \rightarrow \infty} g(x) = 0$. For each g in G , let S_g be the set of all complex functions f with domain $(0, 1]$ for which $f(g)g'$ is simply integrable.

THEOREM 6. (i) *The class S of dominantly integrable functions with domain $(0, 1]$ does not equal any S_g .* (ii) *Given $g \in G$, there is an improperly Riemann integrable function on $(0, 1]$ which is not in S_g .* (iii) *Given any f with domain $(0, 1]$ which is > 0 , continuous and improperly Riemann integrable on $(0, 1]$, $f \in S_g$ for some g in G , and $f \notin S_g$ for some other g in G .*

The next theorem shows that if we drop the condition $g'(x) < 0$ from the definition of G , then part (i) of Theorem 6 becomes false.

THEOREM 7. *There exists a real function $g(x)$ with g' continuous in $[0, \infty)$, $g(0) = 1$, $\lim_{x \rightarrow \infty} g(x) = 0$, and $0 < g(x) \leq 1$ throughout $[0, \infty)$ such that, given any complex function f on $(0, 1]$, $f(g)g'$ is simply integrable if and only if f is dominantly integrable.*

(Not too surprisingly, in light of Theorem 5, the $g(x)$ to be constructed in the proof of Theorem 7 is related to the function e^{-x} , without being monotone.)

II. PROOF OF THEOREMS 1-3

LEMMA 1. *Let f be dominantly integrable, and let $0 < a < b \leq 1$. Then f is Riemann integrable on $[a, b]$.*

Proof of Lemma 1. We shall prove that f is Riemann integrable on $[a, 1]$. As we shall see, it suffices to prove the following statement: For each $\epsilon > 0$ there exists $\delta_1 > 0$ such that if $a = x_0 < x_1 < \dots < x_n = 1$, $x_{j-1}x_j^{-1} > 1 - \delta_1$, $x_{j-1} \leq \xi_j \leq x_j$, and $x_{j-1} \leq \xi'_j \leq x_j$, $j = 1, 2, \dots, n$, then

$$\left| \sum_{j=1}^n [f(\xi_j) - f(\xi'_j)](x_j - x_{j-1}) \right| < \epsilon. \quad (2)$$

Indeed, assume its truth. Let $\delta_2 = a\delta_1$. If $a = x_0 < x_1 < \dots < x_n = 1$, and $x_j - x_{j-1} < \delta_2$, $j = 1, 2, \dots, n$, then $1 - x_{j-1}x_j^{-1} < a^{-1}\delta_2$, and $x_{j-1}x_j^{-1} > 1 - \delta_1$, so we have inequality (2) which is easily seen to imply that $\text{OS}(\text{Re}(f); x_0, x_1, \dots, x_n) \leq \epsilon$, and $\text{OS}(\text{Im}(f); x_0, x_1, \dots, x_n) \leq \epsilon$; hence, $f(t)$ is Riemann integrable on $[a, 1]$. What remains is to prove the above statement.

Choose δ and χ , both in $(0, 1)$, so that any two sums of the type appearing in (1), with $t_0 < \chi$, and every $t_{j-1}t_j^{-1} > 1 - \delta$, differ in absolute value by less than ϵ . Set $\delta_1 = \delta$, and choose a positive integer N such that $(1 - \frac{1}{2}\delta)^N a < \chi$. Set $t_0 = \tau_0 = \tau'_0 = (1 - \frac{1}{2}\delta)^N a, \dots, t_{N-1} = \tau_{N-1} = \tau'_{N-1} = (1 - \frac{1}{2}\delta)a$, and, for $0 \leq j \leq n$, set $t_{N+j} = x_j$, $\tau_{N+j} = \xi_j$, and $\tau'_{N+j} = \xi'_j$.

Then we have

$$\left| \sum_{j=0}^n [f(\xi_j) - f(\xi'_j)](x_j - x_{j-1}) \right| = \left| \sum_{j=0}^{N+n} [f(\tau_j) - f(\tau'_j)](t_j - t_{j-1}) \right| < \epsilon.$$

This completes the proof of Lemma 1.

LEMMA 2. *Let f be dominantly integrable. Then it satisfies RCDI.*

Proof of Lemma 2. Since f is Riemann integrable on each closed subinterval of $(0, 1]$, it is bounded on each such subinterval. Given $\epsilon > 0$, choose δ, χ in $(0, 1)$ such that (1) holds under the conditions following it. Let $0 < t_0 < t_1 < \dots < t_n = 1$, $t_{j-1}t_j^{-1} > 1 - \delta$, $j = 1, 2, \dots, n$. Let $g = \operatorname{Re}(f)$, $h = \operatorname{Im}(f)$. Now $w(f, t_{j-1}, t_j) \leq w(g, t_{j-1}, t_j) + w(h, t_{j-1}, t_j)$; hence, if we show that $\operatorname{OS}(g; t_0, \dots, t_n)$ and $\operatorname{OS}(h; t_0, \dots, t_n)$ are each less than 2ϵ , then it would follow that $\operatorname{OS}(f; t_0, \dots, t_n) < 4\epsilon$. Consider, e.g., $\operatorname{OS}(g; t_0, \dots, t_n)$. Suppose, first, $t_0 < \chi$. If τ_j and τ'_j lie in $[t_{j-1}, t_j]$, $j = 1, \dots, n$, then, using (1),

$$\left| \sum_{j=1}^n (g(\tau_j) - g(\tau'_j))(t_j - t_{j-1}) \right| \leq \left| \sum_{j=1}^n (f(\tau_j) - f(\tau'_j))(t_j - t_{j-1}) \right| < 2\epsilon.$$

Thus $\operatorname{OS}(g; t_0, \dots, t_n) = \sum_{j=1}^n w(g, t_{j-1}, t_j)(t_j - t_{j-1}) \leq 2\epsilon$. If $t_0 \geq \chi$, choose a positive integer N such that $(1 - \frac{1}{2}\delta)^N t_0 < \chi$. Then $\operatorname{OS}(f; t_0, \dots, t_n) \leq \operatorname{OS}(f; (1 - \frac{1}{2}\delta)^N t_0, \dots, (1 - \frac{1}{2}\delta) t_0, t_0, t_1, \dots, t_n) < 4\epsilon$. This proves Lemma 2.

Recall the definition of \hat{f} from the proof of Corollary 2.

LEMMA 3. *Suppose that f satisfies RCDI; then \hat{f} is improperly Riemann integrable on $(0, 1]$.*

Proof of Lemma 3. For a proper δ , $0 < \delta < 1$, and for every positive integer N ,

$$\begin{aligned} & \sum_{j=0}^N [\hat{f}((1 - \tfrac{1}{2}\delta)^{j+1}) - \hat{f}((1 - \tfrac{1}{2}\delta)^j)](1 - \tfrac{1}{2}\delta)^j \\ & \leq \sum_{j=0}^N w(f, (1 - \tfrac{1}{2}\delta)^{j+1}, (1 - \tfrac{1}{2}\delta)^j)(1 - \tfrac{1}{2}\delta)^j \\ & = 2\delta^{-1} \sum_{j=0}^N w(f, (1 - \tfrac{1}{2}\delta)^{j+1}, (1 - \tfrac{1}{2}\delta)^j)((1 - \tfrac{1}{2}\delta)^j - (1 - \tfrac{1}{2}\delta)^{j+1}) < \delta^{-1}, \end{aligned}$$

by RCDI. Since \hat{f} is monotone, it is Riemann integrable on each $[a, 1]$, $0 < a < 1$. Since $\hat{f} \geq 0$, it is improperly Riemann integrable on $(0, 1]$ if

$\int_{\epsilon}^1 f(t) dt$ is bounded for $0 < \epsilon < 1$. But an upper bound for this integral is

$$\left(\sum_{j=0}^{\infty} [f((1 - \frac{1}{2}\delta)^{j+1}) - f((1 - \frac{1}{2}\delta)^j)](1 - \frac{1}{2}\delta)^j \right) + f(1) \leq \delta^{-1} + f(1).$$

LEMMA 4. *Let f satisfy RCDI, and let $0 < a < b \leq 1$. Then f is Riemann integrable on $[a, b]$.*

Proof of Lemma 4. For every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left| \sum_{j=1}^n (f(\tau_j) - f(\tau_j'))(t_j - t_{j-1}) \right| < \epsilon$$

if $a = t_0 < t_1 < \dots < t_n = 1$, and if, for $j = 1, 2, \dots, n$, both τ_j, τ_j' belong to $[t_{j-1}, t_j]$, and $t_{j-1}t_j^{-1} > 1 - \delta$. This, however, implies the Riemann integrability of f on $[a, 1]$; see the proof of Lemma 1.

LEMMA 5. *Suppose f is a complex function, Riemann integrable on each $[a, b] \subset (0, 1]$. For each ϵ, ϵ_1 ($\epsilon > 0, 0 < \epsilon_1 < 1$) there exists δ_1 in $(0, \frac{1}{2})$ such that if $0 < t_0 < \dots < t_n = 1; t_{j-1} \leq \tau_j \leq t_j, t_{j-1}t_j^{-1} > 1 - \delta_1$ ($j = 1, 2, \dots, n$), and $t_{n_1-1} \leq \epsilon_1 < t_{n_1}$ for some $n_1, 1 \leq n_1 \leq n$, then*

$$\left| \sum_{j=n_1}^n f(\tau_j)(t_j - t_{j-1}) - \int_{t_{n_1-1}}^1 f(t) dt \right| < \epsilon.$$

Proof of Lemma 5. We have

$$\begin{aligned} & \left| \sum_{j=n_1}^n f(\tau_j)(t_j - t_{j-1}) - \int_{t_{n_1-1}}^1 f(t) dt \right| \\ & \leq \left| \sum_{j=n_1+1}^n f(\tau_j)(t_j - t_{j-1}) + f(\epsilon_1)(t_{n_1} - \epsilon_1) - \int_{\epsilon_1}^1 f(t) dt \right| \\ & \quad + 3M(t_{n_1} - t_{n_1-1}), \end{aligned}$$

where $M = f((1 - \delta_1)\epsilon_1)$, and $\sum_{j=n_1+1}^n = 0$ if $n_1 = n$. Using the Riemann integrability of f on $[\epsilon_1, 1]$, Lemma 5 follows.

LEMMA 6. *Let a complex function f be Riemann integrable on each $[a, b] \subset (0, 1]$, and let \hat{f} be improperly Riemann integrable on $(0, 1]$; then f is dominantly integrable and improperly Riemann integrable on $(0, 1]$, and $I(f)$ (see Definition 1) is the improper Riemann integral of f on $(0, 1]$.*

In light of Lemmas 2, 3, and 4, Lemma 6 implies Theorems 1 and 2.

Proof of Lemma 6. Since f is improperly Riemann integrable on $(0, 1]$, and f is Riemann integrable on each closed subinterval of $(0, 1]$, it follows that $\int_{0+}^1 |f(t)| dt$ and $\int_{0+}^1 f(t) dt$ exist. Given $\epsilon > 0$, choose ϵ_1 , $0 < \epsilon_1 < 1$, such that $\int_{0+}^{\epsilon_1} f(t) dt < \epsilon$. Using Lemma 5, we see that there exists δ , $0 < \delta < \frac{1}{2}$, such that if $0 < t_0 < \dots < t_n = 1$; $t_{j-1} \leq \tau_j \leq t_j$, $t_{j-1}t_j^{-1} > 1 - \delta$ ($j = 1, \dots, n$), and $t_{n_1-1} \leq \epsilon_1 < t_{n_1}$ for some n_1 , $1 \leq n_1 \leq n$, then

$$\begin{aligned} & \left| \sum_{j=1}^n f(\tau_j)(t_j - t_{j-1}) - \int_{0+}^1 f(t) dt \right| \\ & < \left| \sum_{j=1}^{n_1-1} f(\tau_j)(t_j - t_{j-1}) \right| + \int_{0+}^{\epsilon_1} |f(t)| dt + \epsilon. \end{aligned}$$

Now, under these conditions,

$$\begin{aligned} \left| \sum_{j=1}^{n_1-1} f(\tau_j)(t_j - t_{j-1}) \right| & \leq \sum_{j=1}^{n_1-1} f^{\#}(t_{j-1})(t_j - t_{j-1}) \\ & \leq \sum_{j=1}^{n_1-1} f^{\#}((1 - \delta)t_j)(t_j - t_{j-1}) \\ & \leq \int_{0+}^{\epsilon_1} f^{\#}((1 - \delta)t) dt \\ & \leq (1 - \delta)^{-1} \int_{0+}^{\epsilon_1} f^{\#}(t) dt \\ & \leq 2 \int_{0+}^{\epsilon_1} f^{\#}(t) dt. \end{aligned}$$

Hence

$$\left| \sum_{j=1}^n f(\tau_j)(t_j - t_{j-1}) - \int_{0+}^1 f(t) dt \right| < 3 \int_{0+}^{\epsilon_1} f^{\#}(t) dt + \epsilon < 4\epsilon.$$

LEMMA 7. *Property D on $(0, 1]$ implies dominant integrability.*

The proof of Lemma 7 will complete that of Theorem 3.

Proof of Lemma 7. If there exists a monotone decreasing function $g \geq |f|$ which is improperly Riemann integrable on $(0, 1]$, then $f^{\#} \leq g$ is also improperly Riemann integrable on $(0, 1]$. Thus Lemma 7 follows from Lemma 6.

III. PROOF OF THEOREMS 4 AND 5

Proof of Theorem 4. The first nonimplication was shown in [1, p. 7]. The second nonimplication is exemplified by the function f which is identically zero on $[0, \infty)$ except that $f(n^2) = n^{-2}$, $n = 1, 2, \dots$. The ϵ -variation of this function is $\leq \frac{1}{6}\pi^2$ for each $\epsilon > 0$ (see [1, p. 9]), and the function is improperly Riemann integrable on $[0, \infty)$; hence, by [1, Theorem 3] it is simply (thus absolutely simply) integrable. On the other hand $\hat{f}(x) = \sup_{t \geq x} f(t)$ is not improperly Riemann integrable on $[0, \infty)$, so $f(x)$ does not have property D on $[0, \infty)$.

For a proof of the second implication of the Theorem see [1, Theorem 3 and Example b on p. 9]. The first implication can be proved as follows: Let f be an absolutely simply integrable function. By the first part of Theorem 5, $f(x) \equiv g(e^{-x})e^{-x}$, where g is dominantly integrable. So (with $\hat{g}(t) \equiv \sup_{t \leq x \leq 1} |g(x)|$), $|f(x)| \leq \hat{g}(e^{-x})e^{-x}$ on $[0, \infty)$. Hence if $0 \leq x_0 < x_1 < \dots < x_N < \infty$, $x_j - x_{j-1} \geq \epsilon > 0$, $j = 1, \dots, N$, then (with $x_{N+1} = x_N + \epsilon$),

$$\begin{aligned} \sum_{j=1}^N |f(x_j) - f(x_{j-1})| &\leq 2 \sum_{j=0}^N |f(x_j)| \\ &\leq 2 \sum_{j=0}^N \hat{g}(e^{-x_j}) e^{-x_j} \\ &\leq 2(1 - e^{-\epsilon})^{-1} \sum_{j=0}^N \hat{g}(e^{-x_j})(e^{-x_j} - e^{-x_{j+1}}) \\ &\leq 2(1 - e^{-\epsilon})^{-1} \int_{0+}^1 \hat{g}(t) dt < \infty. \end{aligned}$$

Thus f is of BCV; hence, it is simply integrable [1, Theorem 3].

Proof of Theorem 5. Assuming the equivalence in Theorem 5, the nonimplication there follows by the last part of Theorem 4. Dominant integrability of f implies, by Corollary 1, Theorem 1 and a change of variable, that $|f(e^{-x})|e^{-x}$ is improperly Riemann integrable on $[0, \infty)$. Also if $0 \leq x_0 < x_1 < \dots < x_N < \infty$, and $x_j - x_{j-1} \geq \epsilon > 0$, $j = 1, 2, \dots, N$, then

$$\begin{aligned} \sum_{j=0}^N f(e^{-x_j}) e^{-x_j} &\leq (1 - e^{-\epsilon})^{-1} \left[\sum_{j=1}^N f(e^{-x_{j-1}})(e^{-x_{j-1}} - e^{-x_j}) \right. \\ &\quad \left. + f(e^{-x_N})(e^{-x_N} - e^{-x_{N+\epsilon}}) \right] \leq (1 - e^{-\epsilon})^{-1} \int_{0+}^1 f(t) dt < \infty. \end{aligned}$$

We therefore have

$$\begin{aligned}
 & \sum_{j=1}^N ||f(e^{-x_j})| e^{-x_j} - |f(e^{-x_{j-1}})| e^{-x_{j-1}}| \\
 & \leq \sum_{j=1}^N ||f(e^{-x_j})| - |f(e^{-x_{j-1}})|| e^{-x_j} + |f(e^{-x_{j-1}})| (e^{-x_{j-1}} - e^{-x_j}) \\
 & \leq \sum_{j=1}^N 2\hat{f}(e^{-x_j}) e^{-x_j} + \sum_{j=1}^N \hat{f}(e^{-x_{j-1}}) e^{-x_{j-1}} \\
 & \leq 3 \sum_{j=0}^N \hat{f}(e^{-x_j}) e^{-x_j} \\
 & \leq 3(1 - e^{-\epsilon})^{-1} \int_{0+}^1 \hat{f}(t) dt.
 \end{aligned}$$

Thus, using [3, Theorem 3], dominant integrability of f implies absolute simple integrability of $f(e^{-x}) e^{-x}$.

Assuming absolute simple integrability of $f(e^{-x}) e^{-x}$ we have (by a change of variable): (i) $f(x)$ is Riemann integrable on each $[a, b] \subset (0, 1]$; (ii) the improper Riemann integral $\int_{0+}^1 |f(t)| dt$ exists; and (iii) for each $\delta, 0 < \delta < 1$, and every infinite sequence of positive numbers $1 \geq \xi_0 > \xi_1 > \xi_2 > \dots$ with each $\xi_j \xi_{j-1}^{-1} \leq 1 - \delta$, $\sum_{j=1}^{\infty} ||f(\xi_j)| \xi_j - |f(\xi_{j-1})| \xi_{j-1}| < \infty$. If \hat{f} is improperly Riemann integrable on $(0, 1]$, then f has property D there; hence, f is dominantly integrable by Theorem 3. Therefore, we shall assume that \hat{f} is not improperly Riemann integrable on $(0, 1]$. Choose δ in $(0, 1)$. Then

$$\begin{aligned}
 & \sum_{j=0}^N [\hat{f}((1 - \delta)^{j+1}) - \hat{f}((1 - \delta)^j)](1 - \delta)^j \\
 & \geq -\hat{f}(1) + \sum_{j=0}^{N-1} \hat{f}((1 - \delta)^{j+1})[(1 - \delta)^j - (1 - \delta)^{j+1}]
 \end{aligned}$$

($n = 1, 2, \dots$), so that

$$\sum_{j=0}^{\infty} [\hat{f}((1 - \delta)^{j+1}) - \hat{f}((1 - \delta)^j)](1 - \delta)^j = \infty.$$

For $j = 0, 1, 2, \dots$, pick a τ_j , $(1 - \delta)^{j+1} \leq \tau_j \leq (1 - \delta)^j$, such that $|f(\tau_j)| \geq \frac{1}{2}[\hat{f}((1 - \delta)^{j+1}) - \hat{f}((1 - \delta)^j)]$ (existence of such τ_j becomes evident on considering each of the cases: $\hat{f}((1 - \delta)^{j+1}) = \hat{f}((1 - \delta)^j)$ and $\hat{f}((1 - \delta)^{j+1}) > \hat{f}((1 - \delta)^j)$). Then, for $N = 1, 2, \dots$,

$$\sum_{j=0}^N |f(\tau_j)| \tau_j \geq (1 - \delta) \frac{1}{2} \sum_{j=0}^N [\hat{f}((1 - \delta)^{j+1}) - \hat{f}((1 - \delta)^j)](1 - \delta)^j,$$

so that $\sum_{j=0}^{\infty} |f(\tau_j)| \tau_j = \infty$. Choose k , $0 \leq k \leq 3$, such that $\sum_{j=0}^{\infty} |f(\tau_{4j+k})| \times \tau_{4j+k} = \infty$, and set $\xi_{2j} = \tau_{4j+k}$, $j = 0, 1, 2, \dots$. Next, for $j = 0, 1, 2, \dots$, choose ξ_{2j+1} , $(1 - \delta)^{4j+3+k} \leq \xi_{2j+1} \leq (1 - \delta)^{4j+2+k}$, such that

$$|f(\xi_{2j+1})| \leq 1 + \inf_{(1-\delta)^{4j+3+k} \leq x \leq (1-\delta)^{4j+2+k}} |f(x)|.$$

Let N be an integer ≥ 0 . Then

$$\begin{aligned} & \delta \sum_{j=0}^N |f(\xi_{2j+1})| (1 - \delta)^{4j+2+k} \\ &= \sum_{j=0}^N |f(\xi_{2j+1})| [(1 - \delta)^{4j+2+k} - (1 - \delta)^{4j+3+k}] \\ &\leq \sum_{r=0}^{4N+2+k} \inf_{(1-\delta)^{r+1} \leq x \leq (1-\delta)^r} \{|f(x)| + 1\} [(1 - \delta)^r - (1 - \delta)^{r+1}] \\ &\leq \int_{0+}^1 (|f(t)| + 1) dt; \end{aligned}$$

hence, $\sum_{j=0}^N |f(\xi_{2j+1})| \xi_{2j+1} \leq \delta^{-1} \int_{0+}^1 (|f(t)| + 1) dt$.

Since $\xi_j \xi_{j-1}^{-1} \leq 1 - \delta$, $j = 1, 2, \dots$, we have

$$\begin{aligned} \sum_{j=1}^{2N+2} |f(\xi_j)| \xi_j - |f(\xi_{j-1})| \xi_{j-1} &\geq \sum_{j=1}^{N+1} |f(\xi_{2j})| \xi_{2j} - |f(\xi_{2j-1})| \xi_{2j-1} \\ &\geq \sum_{j=1}^{N+1} |f(\xi_{2j})| \xi_{2j} - \sum_{j=1}^{N+1} |f(\xi_{2j-1})| \xi_{2j-1} \\ &\geq \sum_{j=1}^{N+1} |f(\tau_{4j+k})| \tau_{4j+k} \\ &\quad - \delta^{-1} \int_{0+}^1 (|f(t)| + 1) dt \end{aligned}$$

$\rightarrow \infty$ as $N \rightarrow \infty$, which contradicts (iii) above. This completes the proof of Theorem 5.

IV. PROOF OF THEOREMS 6 AND 7

Proof of Theorem 6. (i) Suppose $S = S_g$ for some g in G . If $f \in S$, then $|f| \in S$. Let h denote the inverse function of g . Then the mapping $F \rightarrow F(h) h'$ maps the set of simply integrable functions onto S . If F , with domain $[0, \infty)$,

is simply integrable, then $-|F(h)h'| \in S$; hence $-|F(h(g))| \cdot |h'(g)| \times (h'(g))^{-1} = |F|$ is simply integrable. Since the simple integral is not an absolute integral [1, p. 7], we have a contradiction.

(ii) Let F be improperly Riemann integrable on $[0, \infty)$ but not simply integrable (e.g., $F(x) \equiv (\sin x)/x$, $F(0) = 1$; [1, p. 9]). Then $F(h)h'$ is improperly Riemann integrable on $(0, 1]$ but $F(h)h' \notin S_g$.

(iii) We shall prove this statement in several steps. \int_0^∞ will denote improper Riemann integral on $[0, \infty)$.

(a) Given p , $0 < p < \infty$, there exists a function F_1 , positive and continuous on $[0, \infty)$, and simply integrable, for which $\int_0^\infty F_1(x) dx = p$; and there exists a function F_2 , positive and continuous on $[0, \infty)$, but not simply integrable, for which $\int_0^\infty F_2(x) dx = p$. Indeed, one can take, e.g., $F_1(x) \equiv 2p/[\pi(x^2 + 1)]$ (cf. [2, p. 931]). As to F_2 , by Theorem 5, it suffices to show that there exists a function f_p , positive, continuous, and improperly Riemann integrable on $(0, 1]$, with $\int_{0+}^1 f_p(x) dx = p$, which is not dominantly integrable. Obviously it is enough to show the existence of f_p for one (any) $p \in (0, \infty)$. Such a function is g , with domain $(0, 1]$, where $g(t) \equiv 1$ off of the intervals $[2^{-n} - 4^{-n}, 2^{-n} + 4^{-n}]$, $n = 2, 3, \dots$; on each interval $[2^{-n}, 2^{-n} + 4^{-n}]$, $n = 2, 3, \dots$, $g(t) \equiv 1 + 2^n - 8^n(t - 2^{-n})$; and on each interval $[2^{-n} - 4^{-n}, 2^{-n})$, $n = 2, 3, \dots$, $g(t) \equiv 1 + 8^n(t - 2^{-n} + 4^{-n})$. Clearly g is positive, continuous, and improperly Riemann integrable on $(0, 1]$. Since (again with $\hat{g}(t) = \sup_{t \leq x \leq 1} |g(x)|$, $0 < t \leq 1$), if $2^{-n-1} < t \leq 2^{-n}$ for some integer $n \geq 2$, then $\hat{g}(t) \geq g(2^{-n}) > 2^n > (2t)^{-1}$, we have $\hat{g}(t) > (2t)^{-1}$ for all t in $(0, 1/4]$, and hence (see Corollary 2 and its proof) g is not dominantly integrable.

(b) Given f_1, f_2 , positive and continuous on $[0, \infty)$, with $\int_0^\infty f_1(x) dx = \int_0^\infty f_2(x) dx$, there exist g_1 and g_2 in G such that $f_1(h_1)h_1' = f_2(h_2)h_2'$, where h_k is the inverse function of g_k ($k = 1, 2$). ($h_1'(1), h_2'(1)$ are left-hand derivatives, $g_1'(0), g_2'(0), (d/dx)[g_1(x)]^{1/2}|_{x=0}$ and $(d/dx)[g_2(x)]^{1/2}|_{x=0}$ are right-hand.)

In fact, for $k = 1, 2$, set

$$g_k(x) \equiv \left[1 - \frac{\int_0^x f_k(t) dt}{\int_0^\infty f_k(t) dt}\right]^2$$

so that, throughout $[0, \infty)$,

$$g_k'(x) = -2 \left[1 - \frac{\int_0^x f_k(t) dt}{\int_0^\infty f_k(t) dt}\right] \frac{f_k(x)}{\int_0^\infty f_k(t) dt},$$

$$\frac{d}{dx} [g_k(x)]^{1/2} = \frac{g_k'(x)}{2[g_k(x)]^{1/2}} = -\frac{f_k(x)}{\int_0^\infty f_k(t) dt}.$$

Hence, throughout $(0, 1]$,

$$\begin{aligned} f_1(h_1(x)) h_1'(x) &= \frac{f_1(h_1(x))}{g_1'(h_1(x))} = - \frac{\int_0^\infty f_1(t) dt}{2[g_1(h_1(x))]^{1/2}} = - \frac{\int_0^\infty f_1(t) dt}{2x^{1/2}} \\ &= - \frac{\int_0^\infty f_2(t) dt}{2x^{1/2}} = f_2(h_2(x)) h_2'(x). \end{aligned}$$

(c) Given F , positive and continuous on $[0, \infty)$, and related to f of (iii) by

$$\int_0^\infty F(x) dx = \int_{0+}^1 f(x) dx,$$

there exists $g \in G$ such that $-f(g(x)) g'(x) = F(x)$ throughout $[0, \infty)$. Indeed, let w be an arbitrary function in G (e.g., $(1+x)^{-1}$), and set

$$f_1 = -f(w) w', \quad f_2 = F.$$

Then $\int_0^\infty f_1(x) dx = \int_0^\infty f_2(x) dx$. By (b), there exist g_1, g_2 in G such that $f_1(h_1) h_1' = f_2(h_2) h_2'$, where h_k is the inverse function of g_k ($k = 1, 2$). Thus, $-f(w(h_1)) w'(h_1) h_1' = F(h_2) h_2'$, and hence, throughout $[0, \infty)$,

$$-f(w(h_1(g_2(x)))) w'(h_1(g_2(x))) h_1'(g_2(x)) g_2'(x) = F(h_2(g_2(x))),$$

namely, $-f(g(x)) g'(x) = F(x)$, where $g = w(h_1(g_2)) \in G$. We can now prove (iii). Set $p = \int_{0+}^1 f(x) dx$. Applying (c) to $F = F_1$ of (a), we obtain that there exists $g \in G$ for which $f \in S_g$. Applying (c) to $F = F_2$ of (a), we get that, for some other $g \in G$, f does not belong to S_g .

Proof of Theorem 7. For $n = 0, 1, \dots$, define $g_1(x)$ on $[7n, 7(n+1))$ as follows. On $[7n, 7n+1]$, $g_1(x) = e^{-x+6n}$. On $[7n+1, 7n+2]$, g_1 is quadratic; $g_1'(7n+1) = -e^{-n-1}$, and $\lim_{x \rightarrow 7n+2-} g_1'(x) = 0$. On $(7n+2, 7n+4]$, $g_1(x) = g_1(2(7n+2) - x)$. On $[7n+4, 7n+5]$, $g_1(x)$ is quadratic; $g_1'(7n+4) = e^{-n}$, and $\lim_{x \rightarrow 7n+5-} g_1'(x) = 0$. On $(7n+5, 7(n+1))$, $g_1(x) = g_1(2(7n+5) - x)$. This defines g_1 on $[0, \infty)$; also $g_1(6) = 1$, and g_1' is continuous in $(0, \infty)$. Set $g(x) \equiv g_1(x+6)$.

Assume that $f(g) g'$ is simply integrable. Then, [1], its ϵ -variation $V(\epsilon)$ is finite for each $\epsilon > 0$.

For $n = 1, 2, \dots$, let

$$S_n = \sup_{\substack{7n \leq x \leq 7n+1 \\ 7n+3 \leq y \leq 7n+4}} |f(g_1(x)) g_1'(x) - f(g_1(y)) g_1'(y)| (\leq V(2) < \infty),$$

and let x_n, y_n satisfy

$$\begin{aligned} 7n &\leq x_n \leq 7n + 1, \\ 7n + 3 &\leq y_n \leq 7n + 4, \\ |f(g_1(x_n)) g_1'(x_n) - f(g_1(y_n)) g_1'(y_n)| &\geq (1/2) S_n \end{aligned}$$

so that

$$\begin{aligned} &|f(g_1(x_n)) g_1'(x_n) - f(g_1(y_n)) g_1'(y_n)| \\ &\geq (1/2) \sup_{7n \leq x \leq 7n+1} |f(g_1(x)) g_1'(x) \\ &\quad - f(g_1(2(7n+2)-x)) g_1'(2(7n+2)-x)| \\ &= \sup_{7n \leq x \leq 7n+1} |f(g_1(x)) g_1'(x)| \geq e^{-n-1} \sup\{|f(x)| : e^{-n-1} \leq x \leq e^{-n}\}. \end{aligned}$$

Set $\xi_n = x_n - 6$, $\eta_n = y_n - 6$, $n = 1, 2, \dots$. Then $1 \leq \xi_1 < \eta_1 < \xi_2 < \eta_2 < \dots$; $\eta_n - \xi_n \geq 2$, $\xi_{n+1} - \eta_n \geq 3$; $n = 1, 2, \dots$. We have

$$\begin{aligned} V(2) &\geq \sum_{n=1}^{\infty} |f(g(\eta_n)) g'(\eta_n) - f(g(\xi_n)) g'(\xi_n)| \\ &= \sum_{n=1}^{\infty} |f(g_1(y_n)) g_1'(y_n) - f(g_1(x_n)) g_1'(x_n)| \\ &\geq \sum_{n=1}^{\infty} e^{-n-1} \sup\{|f(x)| : e^{-n-1} \leq x \leq e^{-n}\}. \end{aligned}$$

Set $s_k = \sup\{|f(x)| : e^{-k-1} \leq x \leq e^{-k}\}$, $k = 1, 2, \dots$. Then for $N = 1, 2, \dots$,

$$\sum_{n=1}^N \left(\sum_{k=1}^n s_k \right) (e^{-n} - e^{-n-1}) = \left(\sum_{n=1}^N s_n e^{-n} \right) - \left(\sum_{k=1}^N s_k \right) e^{-N-1}.$$

Because of the convergence of the last infinite series, $e^{-n-2} s_{n+1} \rightarrow 0$. For $\mu = 0, 1, 2, \dots$, $0 \leq \nu \leq \mu$, let $a_{\mu, \nu} = e^{\nu-\mu}$, so that for every $p \geq 0$, $a_{n, p} \equiv e^{p-n} \rightarrow 0$ as $n \rightarrow \infty$. Also, for every $n \geq 0$,

$$\sum_{k=0}^n |a_{n, k}| = e^{-n}(e^{n+1} - 1)/(e - 1) < e/(e - 1).$$

Hence [9, p. 72] $(\sum_{k=1}^{n+1} s_k) e^{-n-2} \equiv \sum_{k=0}^n e^{k-n} e^{-k-2} s_{k+1} \rightarrow 0$, so that $(\sum_{k=1}^N s_k) e^{-N-1} \rightarrow 0$, and

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^n s_k \right) (e^{-n} - e^{-n-1}) = \sum_{n=1}^{\infty} s_n e^{-n}.$$

Hence

$$\begin{aligned}
 \infty > V(2) &\geq e^{-1} \sum_{n=1}^{\infty} \left(\sum_{k=1}^n s_k \right) (e^{-n} - e^{-n-1}) \\
 &\geq e^{-1} \sum_{n=1}^{\infty} [\sup\{|f(x)| : e^{-n-1} \leq x \leq e^{-n}\}] (e^{-n} - e^{-n-1}) \\
 &\geq e^{-1} \sum_{n=1}^{\infty} [\hat{f}(e^{-n-1}) - \hat{f}(e^{-n})] (e^{-n} - e^{-n-1}).
 \end{aligned}$$

The convergence of $\sum_{n=1}^{\infty} \hat{f}(e^{-n-1})(e^{-n} - e^{-n-1})$ implies that of $\int_{0+}^1 \hat{f}(x) dx$. Thus, f is dominated in $(0, 1]$ by the monotone decreasing function \hat{f} , which is improperly Riemann integrable on $(0, 1]$. By Theorem 3, to show dominant integrability of f it suffices to prove that it is Riemann integrable on each $[e^{-n-1}, e^{-n}]$, $n = 0, 1, \dots$. Choose such an n .

Since $f(g)g'$ is simply integrable, it is improperly Riemann integrable on $[0, \infty)$, and hence, Riemann integrable on $[7n, 7n+1]$. Observe that $g' \neq 0$ throughout $[7n, 7n+1]$. Let h be the inverse function of the restriction of g to $[7n, 7n+1]$, so that $h(e^{-n-1}) = 7n+1$, $h(e^{-n}) = 7n$. Then the substitution $x = h(t)$ in $\int_{7n}^{7n+1} f(g(x))g'(x) dx$ shows the Riemann integrability of f on $[e^{-n-1}, e^{-n}]$.

Conversely, suppose f is dominantly integrable. By Theorem 1, and the fact that g is strictly monotone on each $[k-1, k]$, $k = 1, 2, \dots$, we have, for every $T \geq 1$,

$$\begin{aligned}
 &\left| \int_0^T f(g(x))g'(x) dx + \int_{0+}^1 f(x) dx \right| \\
 &= \left| \left[\sum_{k=1}^{[T]} \int_{k-1}^k f(g(x))g'(x) dx \right] + \int_{[T]}^T f(g(x))g'(x) dx + \int_{0+}^1 f(x) dx \right| \\
 &= \left| - \int_{g(T)}^1 f(x) dx + \int_{0+}^1 f(x) dx \right| \\
 &= \left| \int_{0+}^{g(T)} f(x) dx \right| \rightarrow 0 \quad \text{as } T \rightarrow \infty,
 \end{aligned}$$

since $\lim_{T \rightarrow \infty} g(T) = 0$. ($[T]$ means the largest integer $\leq T$). By [1, Theorem 3], to prove simple integrability of $f(g)g'$ it suffices to show it is of bounded coarse variation.

Let $\epsilon > 0$. We shall show that if $0 \leq \xi_0 < \xi_1 < \dots < \xi_N$, and $\min_{1 \leq r \leq N} (\xi_r - \xi_{r-1}) \geq \epsilon$, then $\Delta = \sum_{r=1}^N |f(g(\xi_r))g'(\xi_r) - f(g(\xi_{r-1}))g'(\xi_{r-1})| \leq$

$14e^2(1 + \epsilon^{-1}) \int_{0+}^{1/e} f(x) dx$ (the integral converging by Corollary 2 and its proof).

Now

$$\Delta \leq 2 \sum_{r=0}^N |f(g(\xi_r)) g'(\xi_r)| = 2 \sum_{n=0}^{\infty} \sum_{\substack{0 \leq r \leq N \\ e^{-n-1} < g(\xi_r) \leq e^{-n}}} |f(g(\xi_r)) g'(\xi_r)|$$

(an "empty" sum means 0). For each $n (= 0, 1, 2, \dots)$ there are seven intervals (or less) of the form $[k-1, k]$ ($k = 1, 2, \dots$) whose union contains every $\xi \geq 0$ with $e^{-n-1} < g(\xi) \leq e^{-n}$; also, for every such ξ , $|g'(\xi)| \leq e^{-n}$. Hence

$$\Delta \leq 2 \sum_{n=0}^{\infty} f(e^{-n-1}) e^{-n} \sum_{\substack{0 \leq r \leq N \\ e^{-n-1} < g(\xi_r) \leq e^{-n}}} 1.$$

For $k = 1, 2, \dots$, there are at most $1 + \epsilon^{-1}$ points ξ_r in $[k-1, k]$. Hence

$$\begin{aligned} \Delta &\leq 14(1 + \epsilon^{-1}) \sum_{n=0}^{\infty} f(e^{-n-1}) e^{-n} \\ &\leq 14e^2(1 + \epsilon^{-1}) \sum_{n=0}^{\infty} f(e^{-n-1})(e^{-n-1} - e^{-n-2}) \\ &\leq 14e^2(1 + \epsilon^{-1}) \int_{0+}^{1/e} f(x) dx. \end{aligned}$$

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